

# Characterization of a noisy quantum process by complementary classical operations

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## Abstract

One of the challenges in quantum information is the demonstration of quantum coherence in the operations of experimental devices. While full quantum process tomography can do the job, it is both cumbersome and unintuitive. In this presentation, we show that a surprisingly detailed and intuitively accessible characterization of errors is possible by measuring the error statistics of only two complementary classical operations of a quantum gate.

## 1 Introduction

Due to rapid technological advances, there is now an increasing number of experimental realizations of multi qubit quantum processes. As more and more complex processes become feasible, it is desirable to develop efficient strategies for the characterization of such gates. Specifically, it may be useful to identify the noise characteristics of a gate without having to perform the large number of measurements necessary for full quantum process tomography.

A particularly promising approach appears to be the analysis of a pair of complementary classical operations performed by the quantum device [1, 2, 3, 4]. The errors observed in such a pair of classical operations provide a surprisingly detailed “map” of the noise in the complete quantum operation. It is therefore possible to use this limited set of measurement results to estimate the performance of the gate in operations that have not been evaluated explicitly.

In the following, we will briefly explain why the coherence of a quantum operation is completely described by the “parallel” performance of a pair of complementary operations, and how the observable error syndromes correspond to the process matrix elements describing the complete quantum process. We then apply our analysis to the actual experimental

data obtained from a recently realized optical quantum controlled-NOT gate. In particular, we can derive estimates for the success of entanglement generation and of Bell state analysis, even though the operations actually measured were completely local.

## 2 Error classification

In general, any noisy quantum process in a  $d$  dimensional Hilbert space can be described using a set of  $d^2$  orthogonal operators  $\hat{\Lambda}_i$  by treating the operators as vectors in an operator space with an inner product defined by the product trace of two operators. If the intended operation is given by the operator  $\hat{U}_0$ , errors may be classified according to deviations from the intended output by multiplying an orthogonal set of output error operators  $\hat{\Lambda}_i$  with the ideal operation  $\hat{U}_0$ . A specific process  $E$  is then described by a process matrix with elements  $\chi_{ij}$ , so that the input-output relation is given by

$$\hat{\rho}_{\text{out}} = E(\hat{\rho}_{\text{in}}) = \sum_{i,j} \chi_{ij} \hat{\Lambda}_i \hat{U}_0 \hat{\rho}_{\text{in}} \hat{U}_0^\dagger \hat{\Lambda}_j. \quad (1)$$

In principle, any choice of operator basis  $\hat{\Lambda}_i$  is allowed. For practical purposes, however,  $\hat{\Lambda}_i$  should be as close to the directly observed errors as possible. For operations on quantum bits, the process should therefore be expanded in terms of the error syndromes given by the Pauli matrices  $I$ ,  $X$ ,  $Y$ , and  $Z$ , acting independently on each qubit. Since quantum information processes are usually defined in the computational basis given by the  $Z$  eigenstates (the  $Z$  basis, for short), it is possible to identify  $X$  with a bit flip error,  $Z$  with a phase error, and  $Y = iXZ$  with a combined bit flip and phase error [5]. However, the symmetry of the operations suggests that it may be more realistic to interpret each operator

as a Bloch vector rotation by  $\pi$  around the appropriate axis. It is then clear that a phase error in the  $Z$  basis will show up as a bit flip in the  $X$  basis, and vice versa. Since experiments on individual qubits can only measure the bit value of a given basis, it is more realistic to regard “phase errors” as observable bit flips in the complementary basis. It is then possible to identify each error operator by a pair of observable error patterns, one in the computational  $Z$  basis and another in the complementary  $Z$  basis.

Table 1: Examples of error indices based on the error patterns observed in the complementary  $Z$  and  $X$  basis.

Error $\hat{\Lambda}_i$	$Z$ error $f_z$	$X$ error $f_x$	index $i$
$I \otimes X \otimes Y$	$011 = 3$	$001 = 1$	3, 1
$Z \otimes Y \otimes Y$	$011 = 3$	$111 = 7$	3, 7
$Y \otimes I \otimes Y$	$101 = 5$	$101 = 5$	5, 5

A complete set of  $d^2 = 2^{2N}$  orthogonal errors  $\hat{\Lambda}_i$  is thus obtained by simply combining the  $2^N$  possible  $N$  qubit errors in  $Z$  with the  $2^N$  possible errors in  $X$ . For convenience, the index  $i$  can then be written as  $i = (f_z, f_x)$ , where  $f_{z/x}$  is the value of the binary number obtained by assigning a digit of 1 to the locations of bit flip errors. Table 1 shows some examples of this labeling system for three qubit errors (64 possibilities).

### 3 Evaluation of experimental results

An experimental quantum gate can now be tested by using only two complementary sets of inputs,  $\{|n\rangle\}$  and  $\{|k\rangle\}$ , chosen in such a way that the correct outputs of the  $d = 2^N$  orthogonal states  $|n\rangle$  are the eigenstates of  $Z$ ,  $\{\hat{U}_0 |n\rangle = |Z_n\rangle\}$ , and the correct outputs of the  $d = 2^N$  orthogonal states  $|k\rangle$  are the eigenstates of  $X$ ,  $\{\hat{U}_0 |k\rangle = |X_k\rangle\}$ . The experimental results can be represented in a pair of error tables showing the probabilities for the various outcomes. Each outcome can then be identified by its error number  $f_{z/x}$ , so that the correct output  $Z_n$  ( $X_k$ ) is given by the conditional probability  $p(0|Z_n)$  ( $p(0|X_k)$ ), and the observed error probabilities are given by

$$\begin{aligned} p(f_z|Z_n) &= \langle Z_n | \hat{\Lambda}_{f_z,0} E(|n\rangle\langle n|) \hat{\Lambda}_{f_z,0} | Z_n \rangle \\ p(f_x|X_k) &= \langle X_k | \hat{\Lambda}_{0,f_x} E(|k\rangle\langle k|) \hat{\Lambda}_{0,f_x} | X_k \rangle. \end{aligned} \quad (2)$$

Outcomes corresponding to the same kind of error can then be averaged to obtain the fidelities  $F_Z$  and  $F_X$  and the error probabilities  $\eta_Z(f_z)$  and  $\eta_X(f_x)$  of the  $Z$  and  $X$  operations, respectively. Specifically, the sums defining the fidelities and the error probabilities read

$$\begin{aligned} F_Z &= \frac{1}{d} \sum_{Z_n=0}^{d-1} p(0|Z_n) & \eta_Z(f_z) &= \frac{1}{d} \sum_{Z_n=0}^{d-1} p(f_z|Z_n) \\ F_X &= \frac{1}{d} \sum_{X_k=0}^{d-1} p(0|X_k) & \eta_X(f_x) &= \frac{1}{d} \sum_{X_k=0}^{d-1} p(f_x|X_k). \end{aligned} \quad (3)$$

Table 2 shows the experimental data reported for an optical quantum controlled-NOT [2] arranged according to the errors  $f_z$  and  $f_x$ . Note that the performance of the experimental device is now described by only two fidelities and  $2d - 2$  error probabilities. For the quantum controlled-NOT and other two qubit operations, this means that only 8 characteristic probabilities are used to evaluate a process fully described by a total of 256 process matrix elements.

Classical intuition already indicates that the probabilities of the errors  $f_z$  observed in  $Z$  should correspond to the sums of the diagonal process matrix elements with the same value of  $f_z$  in the first part of the index  $i$ , and the probabilities observed in  $X$  should correspond to the sums diagonal process matrix elements with the same  $f_x$ . This relation can indeed be confirmed by applying the process matrix definition in eq.(1) to the averages of the conditional probabilities defined by eqs. (2) and (3). As expected, the results read

$$\begin{aligned} F_Z &= \sum_{f_x=0}^{d-1} \chi_{(0,f_x)(0,f_x)}, \quad \eta_Z(f_z) = \sum_{f_x=0}^{d-1} \chi_{(f_z,f_x)(f_z,f_x)}, \\ F_X &= \sum_{f_z=0}^{d-1} \chi_{(f_z,0)(f_z,0)}, \quad \eta_X(f_x) = \sum_{f_z=0}^{d-1} \chi_{(f_z,f_x)(f_z,f_x)}. \end{aligned} \quad (4)$$

It is therefore possible to interpret the diagonal elements  $\chi_{(f_z,f_x)(f_z,f_x)}$  as joint probabilities of the errors  $f_x$  and  $f_z$ , where the unconditional probabilities of  $f_z$  are given by the measurement results  $F_Z$  and  $\eta_Z(f_z)$ , and the unconditional probabilities of  $f_x$  are given by the measurement results  $F_X$  and  $\eta_X(f_x)$ . The process matrix elements can then be estimated by making some additional assumptions about the correlations of errors in  $X$  and in  $Z$ .

The relation between the observed error distributions and the diagonal elements of the density matrix

Table 2: Measurement data for an experimental two qubit gate [2] arranged according to the observed output errors. The error distribution is summarized by the averages at the bottom of each table.

$p(f_z Z_n)$	$f_z = 0$	$f_z = 1$	$f_z = 2$	$f_z = 3$
$Z_n = 00$	0.898	0.031	0.061	0.011
$Z_n = 01$	0.885	0.021	0.088	0.006
$Z_n = 10$	0.819	0.054	0.031	0.096
$Z_n = 11$	0.810	0.099	0.027	0.064
averages	$F_Z = 0.853$	$\eta_Z(1) = 0.051$	$\eta_Z(2) = 0.052$	$\eta_Z(3) = 0.044$

  

$p(f_x X_n)$	$f_x = 0$	$f_x = 1$	$f_x = 2$	$f_x = 3$
$X_k = 00$	0.854	0.044	0.063	0.039
$X_k = 01$	0.870	0.019	0.071	0.040
$X_k = 10$	0.871	0.058	0.050	0.021
$X_k = 11$	0.874	0.013	0.099	0.013
averages	$F_X = 0.867$	$\eta_X(1) = 0.034$	$\eta_X(2) = 0.071$	$\eta_X(3) = 0.028$

can be visualized by arranging the diagonal elements  $\chi_{(f_z, f_x)(f_z, f_x)}$  in a table so that the lines represent the errors  $f_z$  observed in  $Z$  and the columns represent the errors  $f_x$  observed in  $X$ . Table 3 shows this arrangement for a two qubit operation such as the quantum controlled-NOT that was first analyzed by this method in [2]. The experimentally observed fidelities and error probabilities are then given by the sums of the corresponding column or line.

Since all diagonal elements of the process matrix must be positive, the measurement values impose rather strict limitations on various properties of the gate. Of particular interest the estimate of the process fidelity  $\chi_{(0,0)(0,0)} = F_{qp}$ , since it provides a measure of how close the experimental gate is to the intended ideal process defined by  $\hat{U}_0$ . The relation between the process fidelity  $F_{qp}$  and the experimentally observed complementary fidelities  $F_Z$  and  $F_X$  can be determined directly from eq.(4), making use of the fact that the trace of the process matrix is one. The sum of the two complementary fidelities then reads

$$F_Z + F_X = (F_{qp} + 1) - \left( \sum_{f_x=1}^{d-1} \sum_{f_z=1}^{d-1} \chi_{(f_z, f_x)(f_z, f_x)} \right). \quad (5)$$

Since the sum over process matrix elements with both  $f_z \neq 0$  and  $f_x \neq 0$  is always positive, the sum of  $F_Z$  and  $F_X$  can never be larger than the process fidelity plus one. At the same time, eq.(4) ensures that no

classical fidelity is smaller than the process fidelity. For this reason, the two measurement results  $F_Z$  and  $F_X$  limit the process fidelity of an experimental device to the interval given by [1]

$$F_Z + F_X - 1 \leq F_{qp} \leq \text{Min}\{F_Z, F_X\}. \quad (6)$$

The difference between the upper and the lower limit of this interval is equal to  $\text{Max}\{F_Z, F_X\} - 1$ . For devices with at least one very high fidelity, this estimate is therefore nearly as good as a direct measurement of the process fidelity.

In principle, the process fidelity can then be used to estimate important gate properties such as the entanglement capability of the gate [2]. However, more precise information is available if the complete error statistics are used to estimate the fidelities of operations other than the  $Z$  and  $X$  operations observed in the experiment [3, 4]. For this purpose, it is useful to construct a “worst case” error model, where all errors are either pure  $Z$  errors ( $i = (f_z, 0)$ ) or pure  $X$ -errors ( $i = (0, f_x)$ ). The diagonal elements of the process matrix are then given by the corresponding measurement results,

$$\begin{aligned} \chi_{(f_z, 0)(f_z, 0)} &= \eta_Z(f_z), \\ \chi_{(0, f_x)(0, f_x)} &= \eta_X(f_x), \end{aligned} \quad (7)$$

and the process fidelity has its minimal value of  $F_{qp} = F_Z + F_X - 1$ . In the case of the quantum controlled-NOT gate [2], the process matrix elements defined by this noise model are shown in table 4.

Table 3: Illustration of the limits on diagonal process matrix elements defined by the measurement results for the complementary  $Z$  and  $X$  operations for a general 2 qubit operation.

$\chi_{(f_z, f_x)(f_z, f_x)}$	$f_x = 0$	$f_x = 1$	$f_x = 2$	$f_x = 3$	Sum
$f_z = 0$	$F_{qp}$	$\chi_{(0,1)(0,1)}$	$\chi_{(0,2)(0,2)}$	$\chi_{(0,3)(0,3)}$	$F_Z$
$f_z = 1$	$\chi_{(1,0)(1,0)}$	$\chi_{(1,1)(1,1)}$	$\chi_{(1,2)(1,2)}$	$\chi_{(1,3)(1,3)}$	$\eta_Z(1)$
$f_z = 2$	$\chi_{(2,0)(2,0)}$	$\chi_{(2,1)(2,1)}$	$\chi_{(2,2)(2,2)}$	$\chi_{(2,3)(2,3)}$	$\eta_Z(2)$
$f_z = 3$	$\chi_{(3,0)(3,0)}$	$\chi_{(3,1)(3,1)}$	$\chi_{(3,2)(3,2)}$	$\chi_{(3,3)(3,3)}$	$\eta_Z(3)$
Sum	$F_X$	$\eta_X(1)$	$\eta_X(2)$	$\eta_X(3)$	1

## 4 Noise models and fidelity estimates

It is now possible to make predictions for other operations based on this noise model. In particular, any operation resulting in local output states that are eigenstates of some combination of  $X$ ,  $Y$ , and  $Z$  eigenstates has a fidelity given by a well defined sum of  $d = 2^N$  diagonal elements of the process matrix, corresponding to the error operators that stabilize the output states. For example, the two qubit operation resulting in  $ZX$  outputs has a fidelity of

$$F_{zx} = F_{qp} + \chi_{(1,0)(1,0)} + \chi_{(0,2)(0,2)} + \chi_{(1,2)(1,2)} \quad (8)$$

In the case of the quantum controlled-NOT, the  $ZX$  operation is simply the identity operation, since the  $ZX$  eigenstates are also eigenstates of the ideal controlled-NOT operation  $\hat{U}_0$ . We can therefore estimate how well the quantum gate preserves its eigenstates. In terms of the error probabilities  $\eta_{x/z}(F_{x/z})$ , the result reads

$$F_{zx} \geq 1 - \eta_Z(2) - \eta_Z(3) - \eta_X(1) - \eta_X(3) = 0.842. \quad (9)$$

Note that this result is significantly larger than the result of 0.72 defined by the minimal process fidelity.

Another operation of great interest is the generation of entanglement from local  $XZ$  inputs. The ideal operation  $\hat{U}_0$  converts these input states into the four orthogonal Bell states, characterized by their  $XX$ ,  $YY$ , and  $ZZ$  correlations. The fidelity  $F_{E1}$  of this operation is also given by a sum of four process matrix elements, corresponding to the Bell state stabilizers  $II$ ,  $XX$ ,  $YY$ , and  $ZZ$ ,

$$F_{E1} = F_{qp} + \chi_{(3,0)(3,0)} + \chi_{(0,3)(0,3)} + \chi_{(3,3)(3,3)}. \quad (10)$$

The measurement results define a minimal fidelity for this operation given by one minus the single qubit errors  $f_z = 1$ ,  $f_z = 2$ ,  $f_x = 1$  and  $f_x = 2$ ,

$$F_{E1} \geq 1 - \eta_Z(1) - \eta_Z(2) - \eta_X(1) - \eta_X(2) = 0.792. \quad (11)$$

It is therefore possible to improve the estimate of entanglement capability originally reported in [2] without adding any new data.

It may also be interesting to apply the same analysis to the reverse operation, that converts the Bell states into local  $XZ$  eigenstates. The fidelity  $F_{xz}$  of this disentangling operation is given by the process matrix element sum

$$F_{xz} = F_{qp} + \chi_{(2,0)(2,0)} + \chi_{(0,1)(0,1)} + \chi_{(2,1)(2,1)}, \quad (12)$$

and the minimal fidelity defined by the measurement results is

$$F_{xz} \geq 1 - \eta_Z(1) - \eta_Z(3) - \eta_X(2) - \eta_X(3) = 0.806. \quad (13)$$

The minimal fidelity of disentangling the Bell states is thus higher than the minimal fidelity of entanglement generation by the time-reversed process.

Finally, it is also possible to consider the errors in entanglement generation from  $YY$  inputs. The outputs are then maximally entangled states with the stabilizers  $II$ ,  $ZY$ ,  $YX$ , and  $XZ$ . Therefore, the fidelity  $F_{E2}$  is given by

$$F_{E2} = F_{qp} + \chi_{(1,3)(1,3)} + \chi_{(3,2)(3,2)} + \chi_{(2,1)(2,1)}. \quad (14)$$

In this case, all of the stabilizers represent errors that show up in both the  $Z$  and the  $X$  measurements. Therefore, the fidelity  $F_{xz}$  can be as low as the minimal process fidelity,

$$F_{E2} \geq F_Z + F_X - 1 = 0.720. \quad (15)$$

Thus the measurement data provided by the complementary  $Z$  and  $X$  operations provides only a very rough estimate of the operation on  $Y$  inputs.

Of course, the worst case noise estimate given in table 4 is a rather extreme and unlikely interpretation of the measurement data. It may therefore be interesting to compare it with a more realistic estimate based on the assumption that the errors in  $Z$  and

Table 4: “Worst case” estimate of process matrix elements for the experimental data from [2] shown in table 2.

$\chi_{(f_z, f_x), (f_z, f_x)}$	$f_x = 0$	$f_x = 1$	$f_x = 2$	$f_x = 3$	Sum
$f_z = 0$	0.720	0.034	0.071	0.028	0.853
$f_z = 1$	0.051	0	0	0	0.051
$f_z = 2$	0.052	0	0	0	0.052
$f_z = 3$	0.044	0	0	0	0.044
Sum	0.867	0.034	0.071	0.028	1

$X$  are uncorrelated [4]. For this purpose, it is useful to consider the definition of the average fidelity of a quantum process,

$$F_{\text{Av.}} = \int d\Phi \langle \Phi | \hat{U}_0^\dagger E(|\Phi\rangle\langle\Phi|) \hat{U}_0 | \Phi \rangle, \quad (16)$$

where the integral over  $\Phi$  represents a uniform average over all possible states of the  $d$ -dimensional Hilbert space. It has been shown [6] that this average fidelity is related to the process fidelity by

$$F_{\text{Av.}} = \frac{F_{qp} d + 1}{d + 1}. \quad (17)$$

This relation can be explained quite intuitively by using the error analysis above. As eq.(4) indicates, the fidelities for one input basis of  $d$  orthogonal states are given by a sum of the process fidelity  $F_{qp}$  and the diagonal process matrix elements of  $d-1$  errors, out of a total number of  $d^2-1 = (d+1)(d-1)$  possible errors. This means that the probability that the operation on any given input state  $|\Phi\rangle$  is insensitive to any given error is  $(d-1)/(d^2-1) = 1/(d+1)$ . The average over all possible input states  $|\Phi\rangle$  makes this relation exact: the contribution of  $1/(d+1)$  to  $F_{\text{Av.}}$  in eq.(17) simply represents the probability that the errors in the quantum operation result only in an unobservable phase change.

This interpretation of eq.(17) indicates that we can consider the complementary fidelities  $F_X$  and  $F_Z$  as representative fidelities contributing to the average fidelity  $F_{\text{Av.}}$ . If the complementary fidelities are not too different from each other, the most realistic assumption seems to be that the average fidelity  $F_{\text{Av.}}$  is close to the average of the complementary fidelities. An estimate of the most likely value of the process fidelity  $F_{qp}$  can then be obtained from

$$F_{qp}(\text{est.}) \approx \left(1 + \frac{1}{d}\right) \left(\frac{F_Z + F_X}{2}\right) - \frac{1}{d}. \quad (18)$$

Since this estimate implicitly assumes a rather uniform distribution of errors, it is natural to apply

the same assumption to obtain an estimate for the remaining diagonal elements of the process matrix. This means that the joint probabilities of errors given by  $\chi_{(f_z, f_x)(f_z, f_x)}$  should be proportional to the product of the error probabilities  $\eta_Z(f_z)$  and  $\eta_X(f_x)$  for  $f_z, f_x \neq 0$ . With this assumption of uncorrelated errors, the remaining process matrix elements read

$$\begin{aligned} \chi_{(f_z, f_x)(f_z, f_x)}(\text{est.}) &\approx \frac{d-1}{d} \left( \frac{1}{1-F_Z} + \frac{1}{1-F_X} \right) \eta_Z(f_z) \eta_X(f_x) \\ \chi_{(f_z, 0)(f_z, 0)}(\text{est.}) &\approx \left( \frac{d+1}{2d} - \frac{d-1}{2d} \left( \frac{1-F_X}{1-F_Z} \right) \right) \eta_Z(f_z) \\ \chi_{(0, f_x)(0, f_x)}(\text{est.}) &\approx \left( \frac{d+1}{2d} - \frac{d-1}{2d} \left( \frac{1-F_Z}{1-F_X} \right) \right) \eta_X(f_x). \end{aligned} \quad (19)$$

This estimate may be the most realistic description for quantum processes with very similar complementary fidelities  $F_Z$  and  $F_X$ . If the values of  $F_Z$  and  $F_X$  are very different, it is possible that the estimates for  $\chi_{(0, f_x)(0, f_x)}$  or for  $\chi_{(f_z, 0)(f_z, 0)}$  become negative, especially if the higher fidelity is close to one. Although such errors can be corrected by restricting all diagonal elements to positive values, it would be more natural to use a different noise models for such cases.

In the case of our controlled-NOT gate, the similar values of  $F_Z$  and  $F_X$  indicate that the statistical noise model may be appropriate. The corresponding distribution of process matrix elements is shown in table 5. All diagonal elements representing errors now have very similar values, ranging from a minimum of 0.59 % for (0, 3) to a maximum of 1.98 % for (2, 2). The estimated process fidelity of  $F_{qp} = 0.825$  is only a little bit lower than the maximal possible process fidelity of  $\text{Min}\{F_Z, F_X\} = 0.853$ , and much higher than the lower limit of  $F_Z + F_X - 1 = 0.720$ . Statistical considerations thus indicate that the actual process fidelity of the device is likely to significantly exceed the min-

Table 5: Statistical estimate of the process matrix elements for the experimental data from [2] shown in table 2. The assumption of uncorrelated errors results in a uniform distribution of errors over all possibilities.

$\chi(f_z, f_x), (f_z, f_x)$	$f_x = 0$	$f_x = 1$	$f_x = 2$	$f_x = 3$	Sum
$f_z = 0$	0.825	0.0072	0.0150	0.0059	0.853
$f_z = 1$	0.0146	0.0093	0.0194	0.0077	0.051
$f_z = 2$	0.0149	0.0095	0.0198	0.0078	0.052
$f_z = 3$	0.0126	0.0080	0.0168	0.0066	0.044
Sum	0.867	0.034	0.071	0.028	1

imum assumed in the “worst case” estimate shown in table 4.

We can also derive estimates for other operations from the statistical noise model, corresponding to their most likely values. The results derived from eqs. (8, 10,12,14) read

$$\begin{aligned} F_{zx} &\approx 0.874 & F_{E1} &\approx 0.850 \\ F_{xz} &\approx 0.857 & F_{E2} &\approx 0.859. \end{aligned} \quad (20)$$

Naturally, all of these fidelities are now close to the average fidelity of  $F_{\text{av.}} = (F_Z + F_X)/2 = 0.86$ . Interestingly, the estimate for the fidelity  $F_{E2}$  of entanglement generation from  $YY$  inputs is now higher than the estimate for the fidelity  $F_{E1}$  of entanglement generation from  $XZ$  inputs. This change illustrates the fundamental difference between minimal values and likely values in the fidelity estimates. In fact, the two results show that the available data allows a far more precise estimate of  $F_{E1}$  than of  $F_{E2}$ .

## 5 Conclusions

The errors observed in  $N$ -qubit operations can be characterized in terms of the bit flip errors observed in the complementary operations resulting in  $Z$  and in  $X$  output states. It is therefore useful to characterize a device by first measuring the  $2d = 2^{N+1}$  output fidelities of these two complementary operations. Since the only process able to perform both operations with a fidelity of 1 is the intended process  $\hat{U}_0$ , it is possible to derive upper and lower bounds for the process fidelity from these two measurements. For high fidelity processes, this estimate is sufficient to confirm the successful implementation of a multi-qubit gate.

For noisy processes such as the quantum controlled-NOT analyzed above, the details results for the fidelities and errors allow estimates of the process matrix elements corresponding to various noise

models. Even though these error models are not as precise as the estimates obtained from full quantum tomography, it is remarkable that such a detailed analysis is possible using only a small fraction of the  $d^4 = 16^N$  measurement probabilities required for complete quantum process tomography. The evaluation of complementary operations is therefore a particularly efficient method for the characterization of multi-qubit quantum devices.

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